Thermodynamics of Dual CFTs for Kerr-AdS Black Holes

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Recently Gibbons *et al.* in hep-th/0408217 defined a set of conserved quantities for Kerr-AdS black holes with the maximal number of rotation parameters in arbitrary dimension. This set of conserved quantities is defined with respect to a frame which is non-rotating at infinity. On the other hand, there is another set of conserved quantities for Kerr-AdS black holes, defined by Hawking *et al.* in hep-th/9811056, which is measured relative to a frame rotating at infinity. Gibbons *et al.* explicitly showed that the quantities defined by them satisfy the first law of black hole thermodynamics, while those quantities defined by Hawking *et al.* do not obey the first law. In this paper we discuss thermodynamics of dual CFTs to the Kerr-AdS black holes by mapping the bulk thermodynamic quantities to the boundary of the AdS space. We find that thermodynamic quantities of dual CFTs satisfy the first law of thermodynamics and Cardy-Verlinde formula only when these thermodynamic quantities result from the set of bulk quantities given by Hawking *et al.*. We discuss the implication of our results.

I. INTRODUCTION

Black holes in anti-de Sitter (AdS) space have been investigated thoroughly in recent years due to the AdS/CFT correspondence [1]. It was argued by Witten [2] that the thermodynamics of black holes in AdS spaces can be identified with that of dual conformal field theory (CFT) residing on the boundary of the AdS space. Therefore we can get some insights into thermodynamic behavior and phase structure of some strong coupling CFTs by studying corresponding thermodynamics of black holes in AdS space. One of interesting examples of black holes in AdS space is Kerr-AdS black hole, a rotating black hole in AdS space. According to the AdS/CFT correspondence, it was argued by Hawking *et al.* [3] that the thermodynamics of Kerr-AdS black holes can be mapped to that of dual CFTs residing on a rotating Einstein universe. In their paper, the metric of Kerr-AdS black holes with a single rotation parameter in arbitrary dimension was also given. Recently, the metric form of a general Kerr-AdS black hole with the maximal number of rotation parameters in arbitrary dimension has been obtained in [4].

To discuss thermodynamic properties of black holes in AdS space, one has to first calculate the associated conserved charges with the black hole configurations. On the other hand, one wants to know corresponding thermodynamic quantities of dual CFTs via the AdS/CFT correspondence. In the literature, there are different methods to obtain conserved charges for spacetimes which are asymptotically AdS, see, for example, references [5, 6, 7, 8, 9, 10]. In general, all methods give consistent results for static black holes in AdS space. For rotating black holes in AdS space, however, some essential differences appear in the literature. In particular, we would like to mention here that Hawking *et al.* in [3] defined a set of conserved quantities associated with Kerr-AdS black holes and this set of quantities is measured with respect to a frame which is rotating at infinity. On the other hand, very recently Gibbons *et al.* [11] also obtained a set of conserved quantities of Kerr-AdS black holes, measured with a frame which is not-rotating at infinity. And they further showed that the thermodynamic quantities they obtained obey the first law of black hole thermodynamics, while the set of conserved quantities derived by Hawking *et al.* do not satisfy the first law. For other earlier related studies on the Kerr-AdS black holes see [12] and references therein (The first reference in [12] is the first to discuss the first law for the four dimensional Kerr-Newman-AdS black holes).

In this paper we will discuss thermodynamics of dual CFTs associated with Kerr-AdS black holes and clarify the difference between these two sets of conserved charges. By naively mapping the thermodynamic quantities of Kerr-AdS black holes to those of dual CFTs on the boundary of the AdS space, we find that the resulting thermodynamic quantities from those defined by Hawking *et al.* satisfy the first law of thermodynamics and the entropy of CFTs can be expressed in terms of the Cardy-Verlinde formula [13], which is supposed to be an entropy formula of strong coupling CFTs in arbitrary dimension. On the other hand, the resulting thermodynamic quantities from those defined by Gibbons *et al.* do not satisfy the first law of thermodynamics, and the entropy cannot be written in the form of the Cardy-Verlinde form, where a term associated with the pressure of CFTs and volume of the boundary plays an essential role. Our results indicate that in the AdS/CFT correspondence for

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the Kerr-AdS black holes, the thermodynamic quantities of dual CFTs associated with the Kerr-AdS black holes should be obtained by mapping the bulk thermodynamic quantities of Kerr-AdS black holes given by Hawking *et al.*, instead of those by Gibbons *et al.*. This further supports that the dual CFTs to the Kerr-AdS black holes reside on a rotating Einstein universe.

This paper is organized as follows. In the next section, we briefly review two sets of thermodynamic quantities of four dimensional Kerr-AdS black holes, given by Hawking *et al* [3] and by Gibbons *et al.* [11] respectively, and discuss the difference between them. In Sec. III we show that thermodynamic quantities of dual CFTs, by mapping thermodynamic quantities in the definition due to Hawking *et al.* of Kerr-AdS black holes with the maximal number of rotation parameters satisfy the first law of thermodynamics. We then verify in Sec. IV that the entropy of dual CFTs can be expressed in terms of the Cardy-Verlinde formula. We summarize our results in Sec. V with discussions on the implication of our results.

II. CONSERVED CHARGES IN KERR-ADS SPACETIME

The metric of a four dimensional Kerr-AdS black hole can be expressed as

$$
ds^2 = -\frac{\Delta}{\rho^2}(dt - \frac{a}{\Xi}\sin^2\theta d\phi)^2 + \frac{\rho^2 dr^2}{\Delta} + \frac{\rho^2 d\theta^2}{\Delta_\theta}
$$
$$
+ \frac{\Delta_\theta \sin^2\theta}{\rho^2}(adt - \frac{r^2 + a^2}{\Xi}d\phi)^2, \qquad (2.1)
$$

where

$$
\Delta = (r^2 + a^2)(1 + r^2 l^{-2}) - 2mr, \ \Delta_\theta = 1 - a^2 l^{-2}\cos^2\theta,
$$

$$
\rho^2 = r^2 + a^2\cos^2\theta, \quad \Xi = 1 - a^2 l^{-2}. \tag{2.2}
$$

The integration constant m is related to the mass of the black hole and a is the rotation parameter. The black hole horizon r_+ is determined by the equation $\Delta(r)|_{r=r_+} = 0$.

In [11] Gibbons *et al.* obtain the mass and angular momentum associated with the Kerr-AdS black hole by using the background subtraction approach

$$
E = \frac{m}{\Xi^2}, \quad J = \frac{ma}{\Xi^2}. \tag{2.3}
$$

By using Hamiltonian approach, one can also obtain the same mass and angular momentum [6]. The Hawking temperature of the black hole is easily derived by calculating the surface gravity on the horizon

$$
T = \frac{\kappa}{2\pi} = \frac{r_+(1 + a^2 l^{-2} + 3r_+^2 l^{-2} - a^2 r_+^{-2})}{4\pi(r_+^2 + a^2)}, \tag{2.4}
$$

where κ denotes the surface gravity. The entropy of the black hole satisfies the area formula, one quarter of the

horizon area,

$$
S = \frac{1}{4}A = \frac{\pi(r_+^2 + a^2)}{\Xi}, \tag{2.5}
$$

where A is the horizon area. Defining the angular velocity of the black hole with respect to a frame which is non-rotating at infinity [11]

$$
\Omega = \frac{a(1 + r_+^2 l^{-2})}{r_+^2 + a^2}, \tag{2.6}
$$

Gibbons *et al.* [11] explicitly show that these thermodynamic quantities satisfy the first law of black hole thermodynamics

$$
dE = TdS + \Omega dJ. \tag{2.7}
$$

On the other hand, there exist other expressions of conserved charges associated with the Kerr-AdS black hole. For example, Hawking *et al.* give the expression

$$
E' = \frac{m}{\Xi}, \tag{2.8}
$$

for the mass, while still taking $J = ma/\Xi^2$ for the angular momentum. Such a set of conserved charges can also be obtained by using so-called boundary counterterm approach [9], or the Weyl tensor approach suggested by Ashtekar *et al.* [7, 10]. Further they define an angular velocity which is measured relative to a frame rotating at infinity, by

$$
\Omega' \equiv -\frac{g_{t\phi}}{g_{\phi\phi}}\big|_{r=r_+} = \frac{a\Xi}{r_+^2 + a^2}. \tag{2.9}
$$

This differs from Ω by

$$
\Omega - \Omega' = \frac{a}{l^2}. \tag{2.10}
$$

Gibbons *et al.* verify that this set of thermodynamic quantities do not satisfy the first law of black hole thermodynamics

$$
dE' \neq TdS + \Omega' dJ. \tag{2.11}
$$

In addition, they give a relation between E and E'

$$
E = E' + Ja/l^2, \tag{2.12}
$$

which reflects a fact that the conserved charges E and E' are associated with different Killing vectors; the former is related to $(\partial_t + al^{-2}\partial_\phi)$, while the latter to ∂_t.

III. THERMODYNAMICS OF THE DUAL CFT FOR KERR-ADS BLACK HOLES

In the AdS/CFT correspondence, all thermodynamic quantities of dual CFTs can be obtained by mapping bulk ones associated with black holes in AdS space. In

the previous section, we have seen that there exist two sets of conserved charges for the Kerr-AdS black holes: one is related to the Killing vector ∂_t (corresponding to the case adopted by Hawking *et al.*), the other is related to the Killing vector $\partial_t + al^{-2}\partial_\phi$ (corresponding to the case adopted by Gibbons *et al.*). It is then a natural question to ask which set of conserved charges (thermodynamic quantities) of the Kerr-AdS black holes should be mapped to obtain corresponding ones of dual CFTs of the Kerr-AdS black holes. According to the result obtained in the previous section, it looks reasonable to map the set of quantities given by Gibbons *et al.* and to derive thermodynamic quantities of the boundary CFTs, because this set of quantities satisfy the first law of black hole thermodynamics. It turns out it is not correct and we should map the set of quantities given by Hawking *et al.*, since only in this case, resulting thermodynamic quantities of CFTs obey the first law of thermodynamics. In this section we will show this with Kerr-AdS black holes with the maximal number of rotation parameters in arbitrary dimension.

The metric of general Kerr-AdS black holes with the maximal number of rotation parameters in $n \geq 4$ dimensions is [4]

$$
\begin{aligned}
ds^2 &= -W(1+r^2l^{-2})dt^2 + \frac{2m}{U}(Wdt - \sum_{i=1}^{N}\frac{a_i\mu_i^2 d\varphi_i}{\Xi_i})^2 \\
&+ \sum_{i=1}^{N}\frac{r^2+a_i^2}{\Xi_i}\mu_i^2 d\varphi_i^2 + \frac{Udr^2}{V-2m} + \sum_{i=1}^{N+\epsilon}\frac{r^2+a_i^2}{\Xi_i}d\mu_i^2 \\
&- \frac{l^{-2}}{W(1+r^2l^{-2})}(\sum_{i=1}^{N+\epsilon}\frac{r^2+a_i^2}{\Xi_i}\mu_i d\mu_i)^2,
\end{aligned} \tag{3.1}
$$

where

$$
W = \sum_{i=1}^{N+\epsilon}\frac{\mu_i^2}{\Xi_i}, \quad U = r^\epsilon \sum_{i=1}^{N+\epsilon}\frac{\mu_i^2}{r^2+a_i^2}\prod_{j=1}^{N}(r^2+a_j^2),
$$

$$
V = r^{\epsilon-2}(1+r^2l^{-2})\prod_{j=1}^{N}(r^2+a_j^2),
$$

$$
\Xi_i = 1 - a_i^2 l^{-2}. \tag{3.2}
$$

Here a_i stand for $N \equiv [(n-1)/2]$ independent rotation parameters in N orthogonal 2-planes. One has $n = 2N+1$ when n is odd and $n = 2N+2$ when n is even. $\epsilon = (n-1) \bmod 2$ and $n = 2N+1+\epsilon$. In addition, N azimuthal angles ϕ_i and $(N+\epsilon)$ "direction cosines" obeying $\sum_{i=1}^{N+\epsilon}\mu_i^2 = 1$ have been also introduced here.

The outer horizon r_+ of the black hole is determined by the equation $V(r_+) - 2m = 0$. The Hawking temperature T and the entropy S are easily obtained; they are given in [11]

$$
n = odd:
$$

$$
T = \frac{1}{2\pi}\left(r_+(1+r_+^2l^{-2})\sum_{i=1}^{N}\frac{1}{r_+^2+a_i^2} - \frac{1}{r_+}\right),
$$

$$
S = \frac{\omega_{n-2}}{4r_+}\prod_{i=1}^{N}\frac{r_+^2+a_i^2}{\Xi_i}, \tag{3.3}
$$

$$
n = even:
$$

$$
T = \frac{1}{2\pi}\left(r_+(1+r_+^2l^{-2})\sum_{i=1}^{N}\frac{1}{r_+^2+a_i^2} - \frac{1-r_+^2l^{-2}}{2r_+}\right),
$$

$$
S = \frac{\omega_{n-2}}{4}\prod_{i=1}^{N}\frac{r_+^2+a_i^2}{\Xi_i}, \tag{3.4}
$$

where ω_{n-2} denotes the volume of the unit $(n-2)$-sphere:

$$
\omega_{n-2} = \frac{2\pi^{(n-1)/2}}{\Gamma[(n-1)/2]}.
$$

Gibbons *et al.* [11] find that by evaluating Komar integrals, the angular momenta of the black holes associated with the rotation parameters a_i are given by

$$
J_i = \frac{ma_i\omega_{n-2}}{4\pi\Xi_i\Pi_j\Xi_j}. \tag{3.5}
$$

And the energy of the Kerr-AdS black holes is

$$
n = odd: \quad E = \frac{m\omega_{n-2}}{4\pi\Pi_j\Xi_j}\left(\sum_{i=1}^{N}\frac{1}{\Xi_i} - \frac{1}{2}\right), \quad (3.6)
$$

$$
n = even: \quad E = \frac{m\omega_{n-2}}{4\pi\Pi_j\Xi_j}\sum_{i=1}^{N}\frac{1}{\Xi_i}. \tag{3.7}
$$

The angular velocities of the black holes, measured with respect to a frame which is non-rotating at infinity, are

$$
\Omega_i = \frac{(1+r_+^2l^{-2})a_i}{r_+^2+a^2}. \tag{3.8}
$$

These thermodynamic quantities satisfy the first law of black hole thermodynamics

$$
dE = TdS + \sum_{i=1}^{N}\Omega_i dJ_i. \tag{3.9}
$$

As the case in the four dimensional Kerr-AdS black holes, one can obtain another mass (energy) of the black holes associated with the Killing vector ∂_t

$$
E' = \frac{m(n-2)\omega_{n-2}}{8\pi(\prod_{i=1}^{N}\Xi_i)}, \tag{3.10}
$$

and angular velocities of the black holes

$$
\Omega_i' = \frac{a_i\Xi_i}{r_+^2+a_i^2}. \tag{3.11}
$$

They are measured in a frame rotating at infinity. In such a frame, other quantities keep unchanged. Naturally these quantities do not obey the first law of black hole thermodynamics

$$dE' \neq TdS + \sum_{i=1}^{N} \Omega_i' dJ_i, \qquad (3.12)$$

once again.

Next we consider the thermodynamics of dual CFTs to the Kerr-AdS black holes. The dual CFTs reside on the boundary of the bulk metric (3.1) in the spirit of AdS/CFT correspondence. The boundary metric can be determined by using the bulk metric, up to a conformal factor. Consider the four dimensional Kerr-AdS black hole (2.1) as an example. Taking the conformal factor as R/r, where R is a constant with $R \gg l$, we have

$$
\begin{aligned}
ds^2_{CFT} &= \lim_{r \to \infty} \frac{R^2}{r^2} ds_4^2 = \frac{R^2}{l^2}(-dt^2 + \frac{2a \sin^2 \theta}{\Xi} dt d\phi \\
&\quad + \frac{l^2}{\triangle_\theta} d\theta^2 + \frac{l^2 \sin^2 \theta}{\Xi} d\phi^2) \\
&= \frac{R^2}{l^2}(-\frac{\triangle_\theta}{\Xi} dt^2 + \frac{l^2}{\triangle_\theta} d\theta^2 \\
&\quad + \frac{l^2 \sin^2 \theta}{\Xi}(d\phi + al^{-2}dt)^2).
\end{aligned} \qquad (3.13)
$$

Rescaling the time coordinate as

$$Rdt/l = d\tau \qquad (3.14)$$

we reach

$$
\begin{aligned}
ds^2_{CFT} &= -\frac{\triangle_\theta}{\Xi} d\tau^2 + \frac{R^2}{\triangle_\theta} d\theta^2 \\
&\quad + \frac{R^2 \sin^2 \theta}{\Xi}(d\phi + al^{-1}R^{-1}d\tau)^2 \\
&= -d\tau^2 + \frac{2aR \sin^2 \theta}{l\Xi} d\tau d\phi \\
&\quad + \frac{R^2}{\triangle_\theta} d\theta^2 + \frac{R^2 \sin^2 \theta}{\Xi} d\phi^2.
\end{aligned} \qquad (3.15)
$$

Note that the boundary metric (3.15) describes a (2+1)-dimensional rotating Einstein universe with scale factor R. And the spatial volume is $V = 4\pi R^2/\Xi$. Furthermore, we can see that the angular velocities for the rotating Einstein universe are

$$\Omega_\infty = -al^{-2}, \quad \Omega_\infty = -a(lR)^{-1},$$

in the metric (3.13) and (3.15), respectively. For higher dimensional case, the situation is similar to the four dimensional case. That is, the boundary metric where the dual CFTs reside in can be a rotating Einstein universe with scale factor R. We now can obtain the corresponding thermodynamic quantities of dual CFTs by naively mapping quantities associated with the bulk to those of

the dual CFTs on the boundary (3.15). Due to the rescaling relation (3.14), the expressions for the energy, temperature and angular velocity of the dual CFTs have the forms [14] (see also [17, 18, 19, 20])

$$
\begin{aligned}
E_{CFT}(E'_{CFT}) &= \frac{l}{R} E(E'), \quad T_{CFT} = \frac{l}{R} T, \\
\Omega_{CFT}(\Omega'_{CFT}) &= \frac{l}{R} \Omega(\Omega').
\end{aligned} \qquad (3.16)
$$

In the mapping process, the angular momentum J and entropy S remain unchanged:

$$J_{CFT} = J, \quad S_{CFT} = S, \qquad (3.17)$$

since entropy stands for the degrees of freedom of system and the angular momentum is conserved charge associated with the Killing vector $\partial/\partial\phi$. Note that here our rescaling relations for the angular velocity (3.16) and angular momentum (3.17) are different from those in [19]. We think our rescaling relations are correct. In addition, let us mention that since we are considering $(n-1)$-dimensional CFTs which reside on the boundary with volume V, the pressure of the CFTs has a simple relation to the energy E and volume:

$$P = \frac{E}{(n-2)V}. \qquad (3.18)$$

Now we consider the mapping of the set of thermodynamic quantities in the prescription given by Hawking *et al.*.

i) when $n = odd$, one has $\epsilon = 0$, and

$$
\begin{aligned}
E'_{CFT} &= \frac{ml(n-2)\omega_{n-2}}{8\pi R(\Pi_i \Xi_i)}, \quad J_{iCFT} = \frac{ma_i \omega_{n-2}}{4\pi \Xi_i(\Pi_j \Xi_j)}, \\
T_{CFT} &= \frac{l}{2\pi R}\left(r_+(1 + r_+^2 l^{-2})\sum_{i=1}^{N} \frac{1}{r_+^2 + a_i^2} - \frac{1}{r_+}\right), \\
S &= \frac{\omega_{n-2}}{4r_+}\prod_{i=1}^{N} \frac{r_+^2 + a_i^2}{\Xi_i}, \quad \Omega_{iCFT} = \frac{la_i \Xi_i}{R(r_+^2 + a_i^2)}
\end{aligned} \qquad (3.19)
$$

The spatial volume of the boundary

$$V = \frac{\omega_{n-2}R^{2N-1}}{\Pi_i \Xi_i}, \qquad (3.20)$$

and then the pressure of the CFTs

$$P' = \frac{E'_{CFT}}{(n-2)V}. \qquad (3.21)$$

ii) when $n = even$, one has $\epsilon = 1$, and

$$
\begin{aligned}
E'_{CFT} &= \frac{ml(n-2)\omega_{n-2}}{8\pi R(\Pi_i \Xi_i)}, \quad J_{iCFT} = \frac{ma_i \omega_{n-2}}{4\pi \Xi_i(\Pi_j \Xi_j)}, \\
T_{CFT} &= \frac{l}{2\pi R}\left(r_+(1 + r_+^2 l^{-2})\sum_{i=1}^{N} \frac{1}{r_+^2 + a_i^2} - \frac{1 - r_+^2 l^{-2}}{2r_+}\right), \\
S &= \frac{\omega_{n-2}}{4}\prod_{i=1}^{N} \frac{r_+^2 + a_i^2}{\Xi_i}, \quad \Omega'_{iCFT} = \frac{la_i \Xi_i}{R(r_+^2 + a_i^2)}.
\end{aligned} \qquad (3.22)
$$

In this case, the volume V and pressure of the CFTs are

$$V = \frac{\omega_{n-2}R^{2N}}{\Pi_i \Xi_i}, \quad P' = \frac{E'_{CFT}}{(n-2)V}, \qquad (3.23)$$

respectively. We find that these thermodynamic quantities satisfy the first law of thermodynamics

$$dE'_{CFT} = T_{CFT}dS + \sum_{i=1}^{N} \Omega'_{iCFT}dJ_i - P'dV. \quad (3.24)$$

Its validity from $n = 4$ up to $n = 11$ has been checked by using *Mathematica*. Note that here the scale factor R in (3.24) is variable. When considering R as a constant and setting $R = l$, we can see from (3.16) and (3.17) that all quantities for the dual CFTs reduce to corresponding ones for Kerr-AdS black holes. Indeed, in this case, the first law (3.24) for the dual CFTs is found to degenerate to the first law (3.9) for the black holes.

On the other hand, if we map the set of thermodynamic quantities given by Gibbons *et al.* according to the relation (3.16) and (3.17), resulting thermodynamic quantities of dual CFTs do not satisfy the first law of thermodynamics

$$dE_{CFT} \neq T_{CFT}dS + \sum_{i=1}^{N} \Omega_{iCFT}dJ_i - PdV. \quad (3.25)$$

Therefore we conclude that the thermodynamic quantities of dual CFTs to the Kerr-AdS black holes satisfy the first law of thermodynamics if we map the set of bulk thermodynamic quantities given in the prescription of Hawking *et al.*, instead they do not if we map the set of bulk thermodynamic quantities given by Gibbons *et al.*. The results can explained as follows. From the metric (3.15), we can see that the dual CFTs reside on a rotating Einstein universe, where the timelike Killing vector is $\partial/\partial\tau$. Therefore the energy of the CFTS should be measured respect to this Killing vector. In other words, all physical quantities should be measured in the rotating frame. Our results also further confirm the argument by Hawking et al. [3] that the dual CFTs of Kerr-AdS black holes resides in a rotating Einstein universe. In addition, from the result given by Gibbons *et al.* it seems to tell us that black hole thermodynamics should be measured in a frame which is static at infinity, or with respect to static observers at infinity.

IV. CARDY-VERLINDE FORMULA FOR KERR-ADS BLACK HOLES

In order to further verify that thermodynamic quantities of dual CFTs to the Kerr-AdS black holes should be obtained by mapping the set of bulk thermodynamic quantities given in the prescription of Hawking *et al.*, in this section, we will show that only the thermodynamic quantities of dual CFTs by mapping the set of

bulk thermodynamic quantities given in the prescription of Hawking *et al.* obey the Cardy-Verlinde formula [13], and resulting thermodynamic quantities from the set of bulk quantities given by Gibbons *et al.* do not satisfy the Cardy-Verlinde formula.

There is a well-known entropy formula for a $(1+1)$-dimensional conformal field theory, namely, the Cardy formula [15]. In an elegant paper [13], in the spirit of AdS/CFT correspondence, Verlinde argued that there is a similar entropy formula for CFTs in higher dimensions. The formula "derived" by Verlinde is called Cardy-Verlinde formula in the literature. Indeed this formula has been checked to hold for various CFTs with AdS gravity duals, such as Schwarzschild-AdS black holes [13], Kerr-AdS black holes [16], Hyperbolic and charged black holes [17], Taub-Bolt-AdS instanton [18], Kerr-Newmann-AdS black holes [19] and so on (see also [20] and references therein). In addition, it has been found that entropies of black hole horizons [21] and cosmological horizons [22] in asymptotically dS spaces can also be expressed in terms of the Cardy-Verlinde formula. The Cary-Verlinde formula is supposed to be an entropy formula for CFTs in arbitrary dimension.

Consider a CFT living in $(n-1)$-dimensional spacetime described by the metric

$$ds^2 = -dt^2 + R^2 d\Omega_{n-2}^2, \qquad (4.1)$$

where R is the radius of an $(n-2)$-dimensional sphere. The Cardy-Verlinde formula can be expressed as

$$S = \frac{2\pi R}{n-2}\sqrt{E_c(2E - E_c)}, \qquad (4.2)$$

where E_c denotes the Casimir energy, non-extensive part of the energy E of CFT, defined as

$$E_c \equiv (n-2)(E+PV-TS) = (n-1)E-(n-2)TS. \quad (4.3)$$

When some chemical potentials appear in some CFTs, more terms associated with these chemical potentials should be included to the definition (4.3). For example, for the case of rotating black holes, E_c in (4.3) should be modified to $E_c \equiv (n-2)(E + PV - TS - \Omega J)$.

With the thermodynamic quantities of dual CFTs given in the previous section, we can see that the Cardy-Verlinde formula holds if we map the set of bulk thermodynamic quantities given in the prescription of Hawking *et al.*. Here the Casimir energy E_c is defined as

$$\begin{aligned} E_c &\equiv (n-2)(E'_{CFT} + P'V - T_{CFT}S - \sum_{i=1}^{N}\Omega'_{iCFT}J_i) \\ &= (n-1)E_{CFT} - (n-2)T_{CFT}S \\ &\quad -(n-2)\sum_{i=1}^{N}\Omega'_{iCFT}J_i). \end{aligned} \qquad (4.4)$$

A straightforward calculation gives

i) when $n = odd$,

$$E_c = \frac{(n-2)l\omega_{n-2}}{8\pi R r_+^2} \prod_{i=1}^{N} \frac{r_+^2 + a_i^2}{\Xi_i},$$

$$2E'_{CFT} - E_c = \frac{(n-2)\omega_{n-2}}{8\pi l R} \prod_{i=1}^{N} \frac{r_+^2 + a_i^2}{\Xi_i},$$

$$\frac{2\pi R}{n-2} \sqrt{E_c(2E'_{CFT} - E_c)} = \frac{\omega_{n-2}}{4r_+} \prod_{i=1}^{N} \frac{r_+^2 + a_i^2}{\Xi_i}$$
$$= S. \qquad (4.5)$$

ii) when $n = even$,

$$E_c = \frac{(n-2)l\omega_{n-2}}{8\pi R r_+} \prod_{i=1}^{N} \frac{r_+^2 + a_i^2}{\Xi_i},$$

$$2E'_{CFT} - E_c = \frac{(n-2)\omega_{n-2}r_+}{8\pi l R} \prod_{i=1}^{N} \frac{r_+^2 + a_i^2}{\Xi_i},$$

$$\frac{2\pi R}{n-2} \sqrt{E_c(2E'_{CFT} - E_c)} = \frac{\omega_{n-2}}{4} \prod_{i=1}^{N} \frac{r_+^2 + a_i^2}{\Xi_i}$$
$$= S. \qquad (4.6)$$

Thus we have explicitly checked that the Cardy-Verlinde formula holds for the dual CFTs to the Kerr-AdS black holes in arbitrary dimension, if resulting thermodynamic quantities are obtained by mapping the set of bulk thermodynamic quantities in the prescription of Hawking *et al.*. On the other hand, if we replace E'_{CFT} by E_{CFT}, Ω'_{iCFT} by Ω_{iCFT}, and P' by P in (4.4), it is easy to check that the entropy of dual CFTs cannot be recast into the Cardy-Verlinde form

$$S \neq \frac{2\pi R}{n-2} \sqrt{E_c(2E_{CFT} - E_c)}. \qquad (4.7)$$

Here it might be worth mentioning again that the dual CFTs of the Kerr-AdS black holes do not reside in the spacetime (4.1), a static sphere, but on a rotating Einstein universe (sphere) like (3.15).

V. CONCLUSION AND DISCUSSION

For the Kerr-AdS black holes in arbitrary dimension, there exist two sets of thermodynamic quantities: one is given by Hawking *et al.* [3], and is measured in a frame rotating at infinity; the other is given by Gibbons *et al.* [11], which is measured in a frame which is non-rotating at infinity. And very recently Gibbons *et al.* have shown that only the set of thermodynamic quantities given by them satisfy the first law of black hole thermodynamics. In this paper we have investigated the thermodynamics of dual CFTs to the Kerr-AdS black holes with the maximal number of rotation parameters. We have found that thermodynamic quantities of dual CFTs resulting from the set of bulk thermodynamic quantities given in the prescription given by Hawking *et al.* satisfy the first law of thermodynamics and Cardy-Verlinde formula, which is supposed to be an entropy formula of CFTs in higher dimensions, instead the thermodynamic quantities of dual CFTs obtained by mapping the set of bulk thermodynamic quantities given by Gibbons *et al.* do not obey the first law of thermodynamics and cannot be rewritten in terms of the Cardy-Verlinde formula.

Our results further indicate that the dual CFTs to the Kerr-AdS black holes reside in a boundary which is a rotating Einstein universe. Thermodynamic quantities of dual CFTs should be obtained by mapping the set of bulk thermodynamic quantities given in the prescription of Hawking *et al.*, although this set of bulk thermodynamic quantities do not obey the first law of black hole thermodynamics. On the other hand, from the result obtained by Gibbons *et al.* one can see that black hole thermodynamics should seemingly be measured with respect to static observers at infinity.

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